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Prediction of Turbulent Shear Layers in Turbomachines

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Turbulent shear layers in turbomachines differ from turbulent boundary layers on airfoils in at least seven important respects.

- (1) Stronger three-dimensional effects
- (2) High rates of heat transfer at comparatively low Mach numbers
- (3) Larger camber
- (4) Stronger accelerations and changes of direction
- (5) Lower Reynolds number
- (6) High free-stream turbulence
- (7) Interaction of two shear layers

We have been working on several of these problems as part of an exploration of the limits of boundary-layer theory. The objects are to extend boundary-layer prediction methods to the special cases that we call real life and to use these special cases to test hypotheses used in simpler flows more severely than existing experimental data for the simpler flows permit. Results of the work are presented as part of a discussion of turbomachine problems.

Most present-day prediction methods for turbulent flow, such as those discussed at the 1968 Stanford meeting (ref. 1), refer to rather idealized cases, although several of the methods have been extended to compressible or three-dimensional flow to make them more directly useful to the aircraft industry. Moderate three-dimensionality of the mean flow does not seem to have much effect on the behavior of the turbulence, which is itself always three-dimensional, and there is now adequate evidence that the effects of compressibility on turbulence are small if the density fluctuation is a small fraction of the mean density (but see ref. 1a).

In turbomachine boundary layers, three-dimensionality is more pronounced and density fluctuations due to heat transfer can be much larger than those due to high Mach number. These are obvious and important effects; however, there are several more subtle phenomena to be found in turbomachines and I believe that we ought to take notice of them even at

this early stage. The list of "special effects" given in the abstract is obvious enough—the subtlety is in the way they modify the behavior of the turbulence.

To show that I am not wasting your time with academic trivia, table I shows the conditions under which some of these special effects produce a 10-percent change in surface shear stress or in distance to separation.

The amount of heat transfer or three-dimensionality needed is rather large, but the quoted values of camber, Reynolds number, and free-stream turbulence are typical, or even conservative, figures for turbomachines. Several of these less obvious special effects may occur simultaneously; with good luck they may cancel—with bad luck they will not.

A few recent references have been inserted in this published version, but the text is otherwise that presented at the symposium.

TABLE I.—*Strength of Special Effects Needed to Change Surface Shear Stress or Distance to Separation by 10 Percent*

Special effect	Order of magnitude
Sweepback (with given chordwise pressure gradient).....	$= 45 \text{ deg}$
Heat transfer.....	$T_w/T = 0.7$
Longitudinal curvature (camber).....	$\delta/R = 1/80$ or 35 degrees turning angle
Rotation (component about spanwise axis).....	$\Omega\delta/U = 1/80$
Low Reynolds number (C_f compared with Schoenherr value).....	$U_\infty\theta/\nu = 650$ ($U_\infty x/\nu \ 3 \times 10^5$)
Free stream turbulence (small scale).....	3%
Free stream unsteadiness (large scale: $\omega\delta/2\pi U = 0.1$).....	30% (changes mean C_f by 10 percent) 3.5% (minimum C_f 10 percent below mean C_f)

THREE-DIMENSIONAL BOUNDARY LAYERS

If we postpone discussion of flows near streamwise corners and edges where several different Reynolds stress components contribute to the acceleration of the fluid, we can pose the problem of three-dimensional boundary layers as, "What is the direction of the Reynolds shear stress?" People who use mixing-length or eddy-viscosity concepts in three-dimensional flow implicitly assume that the answer is, "The same as the direction of the mean shear" (components $\partial U/\partial y$ and $\partial W/\partial y$, where y is the direction normal to the surface). Figure 1 shows the directions of the velocity, shear stress, and mean shear in a mildly three-dimensional boundary layer with about 3.5 degrees of crossflow (ref. 2) relaxing back to a two-dimensional state. Note that the *difference* between the directions of shear stress and of mean shear is comparable with the mean crossflow angle, except near the surface where the flow is in local equilibrium and the mixing-length formula, with $l=Ky$, is expected to hold. This mild three-dimensional flow can be predicted to within the rather

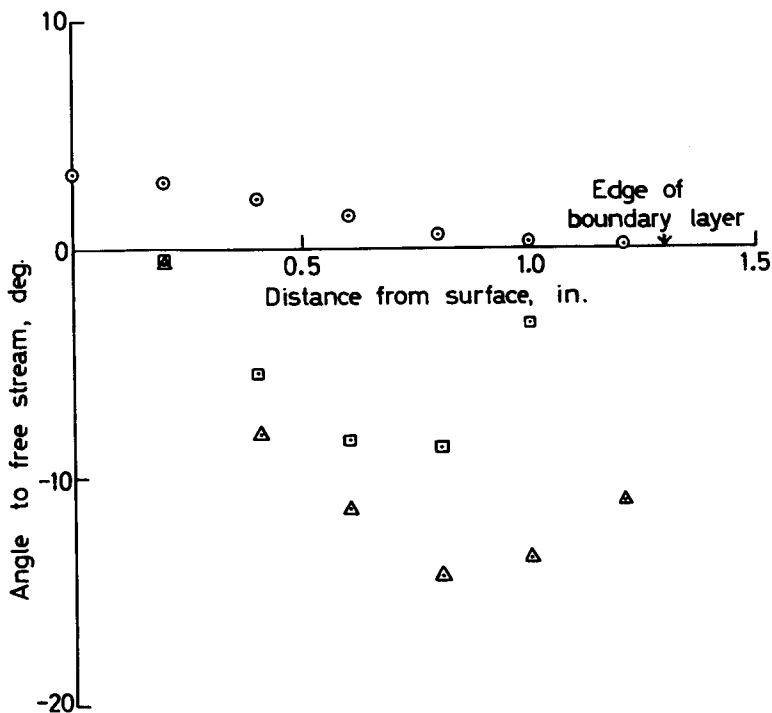


FIGURE 1.—Measured directions of velocity vector (circled points), shear stress (squared points), and mean shear (triangular points) in the boundary layer on a 45-degree swept wing.

poor experimental accuracy by an extension (ref. 3) of the boundary-layer calculation method we developed at the NPL (See fig. 2; the calculations are somewhat more plausible than the experiments.)

A more strongly three-dimensional flow investigated by J. P. Johnston (ref. 4) shows some very curious effects: the boundary layer is initially two-dimensional, but on application of a strong pressure gradient the shear stress vector, far from following the mean shear vector, actually yaws in the opposite direction. It is very hard to reconcile this with any of our current ideas about turbulence, but the hot-wire measurements of shear stress are not infallible. All we can safely conclude is that the prediction of three-dimensional effects stronger than those normally occurring on a moderately swept wing must be treated with caution. For practical purposes, many two-dimensional or three-dimensional flows subjected to sudden pressure changes can be predicted by using the mixing-length formula in the inner layer and Bernoulli's equation in the outer layer.

EFFECTS OF STREAMLINE CURVATURE ON TURBULENCE

Several experiments (refs. 5, 6, and 7; see also the paper by J. P. Johnston in this session, and for a recent review see ref. 7a) have shown that longitudinal surface curvature, or a component of rotation in the direction of the mean vorticity vector, can have a large effect on turbulence, quite apart from any extra terms that may appear in the mean-motion equations. Since Professor Johnston is dealing with the case of rotation, I will confine myself to curvature effects. Roughly, the analysis for one can be applied to the other by reading Ω for U/R . Highly cambered airfoils, particularly turbomachine blades, can suffer appreciably from curvature effects. ("Suffer" is the word, because turbulent shear stress is reduced on convex surfaces, leading to premature upper-surface separation, and increased on concave surfaces, leading to greater lower-surface drag and heat transfer.)

I drew a first-order analogy between the effect of centrifugal or Coriolis forces and the effect of buoyancy, relying on the experimental fact that the correlation between the velocity fluctuation u and the density fluctuation ρ' in a heated shear flow is very strong, so that there should also be a strong correlation between the separate effects of the fluctuating centrifugal force $2Uu/R$ and the fluctuating buoyancy force $-g\rho'$. A first-order formula for the effect of curvature, suggested in reference 8 on the basis of the Monin-Oboukhov meteorological formula, is

$$l/l_{Ri=0} = 1 - \beta Ri \quad (1)$$

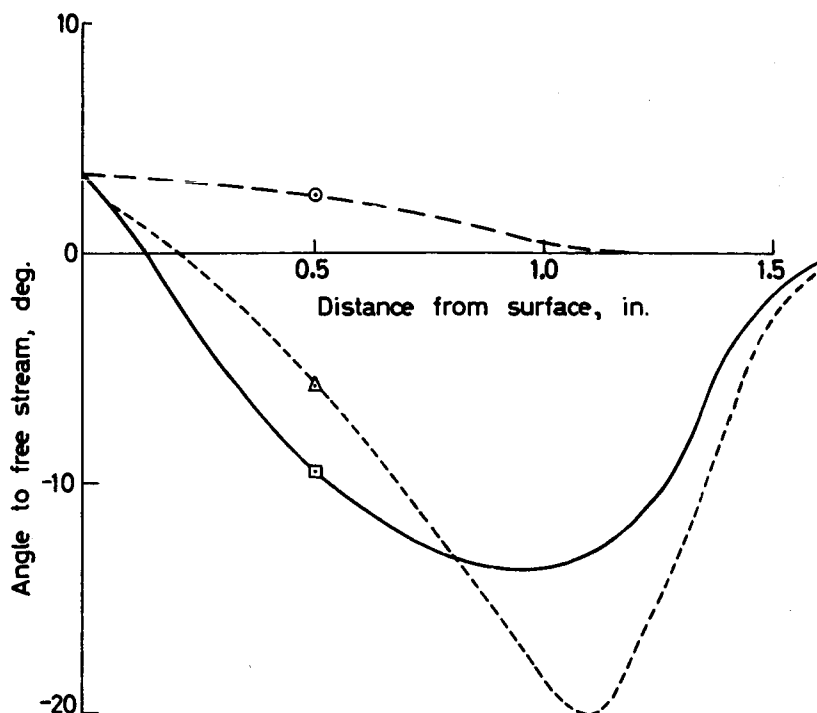


FIGURE 2.—Calculations for the boundary layer of figure 1 (same notation).

where the "apparent mixing length," l , is defined as $\sqrt{(\tau/\rho)}/(\partial U/\partial y)$ and is not ascribed any physical significance; $Ri = 2(U/R)/(\partial U/\partial y)$ is a first approximation to an equivalent Richardson number; and β is a constant, equal to about 7 on a convex surface ($Ri > 0$, stable conditions) and about 4 on a concave surface (unstable conditions). Typically, in the outer part of a turbulent boundary layer on a convex surface, $l/l_0 = 1 - 40\delta/R$. Thus, taken at face value, the analogy suggests that turbulence might die away altogether ("relaminarization") at values of δ/R typical of highly cambered blades; however, this crude analogy can scarcely be expected to work if the turbulence structure is radically changed by strong body forces, and we need something better for turbomachine blades. Relaminarization has been observed by Halleen and Johnston (ref. 6) in a rotating flow; also, Patel (ref. 4) commented that his velocity profiles on a highly convex surface were similar to those he found in relaminarization (ref. 10), but this effect can be explained by the Monin-Oboukhov formula.

In a study of rapid distortion of turbulent shear flow, my doctoral student, Mr. I. P. Castro, has made some measurements in the mixing

layer of an impinging jet. It can be seen from figure 3 that the growth of the shear layer is retarded by strong (stabilizing) curvature. The quantity plotted is a rather arbitrary geometrical width (corrected for slight three-dimensional effects) and not a true mass flow, so that its behavior in the region of strong distortion should not be taken too seriously. The surprising thing is that the growth rate returns to normal rather quickly. (There is a hint of an *overshoot* in growth rate, so that things may be more complicated than they seem; see Discussion and ref. 10a.)

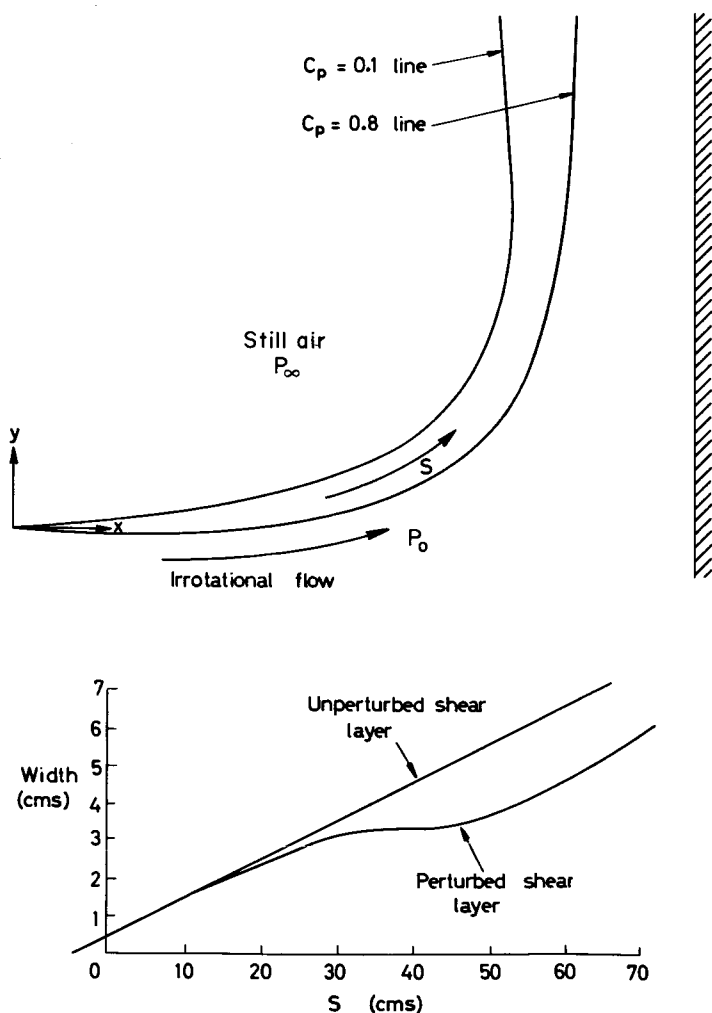


FIGURE 3.—Flow in deflected mixing layer. C_p is total-pressure coefficient, $(P-p_\infty)/P_0-p_\infty$; "Width" is distance between points where C_p is 0.8 and 0.1.

BOUNDARY LAYERS AT LOW REYNOLDS NUMBERS

Coles (ref. 11) has shown, by a painstaking analysis of data, that the velocity defect law in a constant-pressure turbulent boundary layer, usually written as

$$\frac{U_{\infty} - U}{u_{\tau}} = f\left(\frac{y}{\delta}\right) \quad (2)$$

depends on Reynolds number if $U_{\infty}\theta/\nu < 5000$. (Roughly, $U_{\infty}x/\nu = 2.5 \times 10^6$, a high Reynolds number by turbomachine standards.) There remains a small probability that this "Reynolds number" effect may, in fact, be caused by disturbances arising in the transition region but, in any event, Coles' correlation seems to be a universal one, valid for different transition positions and transition devices.

Coles' analysis relies on the constancy of K in the "mixing-length" formula in the inner layer

$$\partial U / \partial y = \sqrt{(\tau / \rho)} / Ky \quad (3)$$

or, more specifically, on the constancy of K and A in the logarithmic velocity profile

$$\frac{U}{u_{\tau}} = \frac{1}{K} \left[\log \left(\frac{u_{\tau} y}{\nu} \right) + A \right] \quad (4)$$

Recently, Simpson (ref. 12) has suggested, on the basis of his measurements in transpired boundary layers, that K may be a function of Reynolds number. Simpson's results could be explained almost equally well in terms of Coles' suggested defect-law behavior; however, incontrovertible evidence of Reynolds number effects on the inner layer velocity profile in pipe and duct flow has been presented by Patel and Head (ref. 10) and merits some discussion.

The changes in the logarithmic law found by Patel and Head can be correlated in terms of an inner-layer parameter, the dimensionless shear stress gradient, $\partial \tau^+ / \partial y^+ = (\nu / \rho u_{\tau}^3) \partial \tau / \partial y$, which has been used by several workers to correlate relaminarization effects in accelerated flows. Even at the lowest Reynolds numbers at which turbulent flow is possible, the values of $\partial \tau^+ / \partial y^+$ found in a *constant-pressure* boundary layer are very much smaller than those associated with inner-layer changes in pipe flow. Therefore, Patel and Head's measurements actually contradict Simpson's suggestion and imply that the inner layer in a low-Reynolds-number boundary layer follows the usual logarithmic law if the pressure gradient is small. It follows that the defect-law changes observed by Coles are probably real and necessarily caused by viscous effects in the outer region of the boundary layer. In order to predict low-Reynolds-number boundary layers we need more data on these viscous effects, although both Herring

and Mellor (ref. 13) and I (unpublished) have inserted an empirical Reynolds number dependence into the specification of shear stress in the outer layer. (The change in total shear stress is *many times* larger than the viscous shear stress in the outer layer.)

The photographs of a smoke-filled boundary layer at low Reynolds number published by Fiedler and Head (ref. 14) suggest that the interface between the turbulent and nonturbulent fluid has more large-scale irregularities than at high Reynolds number. There is no evidence for instability of the viscous superlayer but, since the defect law in a pipe or duct seems to be independent of Reynolds number everywhere outside the viscous sublayer, it seems that the Reynolds-number dependence of the defect law in a boundary layer must be associated with the presence of a free boundary.

In the analysis of Patel and Head's measurements mentioned above (carried out by Dr. G. D. Huffman of Allison while in our department, ref. 16a) it was found that the detailed profiles could be well represented by the mixing-length formula (eq. (3)) with the van Driest mixing-length specification

$$1 = Ky[1 - \exp(-\sqrt{\tau/\rho} y/\nu A^+)] \quad (5)$$

with A^+ a function of $\partial\tau^+/\partial y^+$, chosen to optimize the fit. This seems to be a satisfying demonstration of the wide applicability of inner-layer similarity ideas and strongly supports the view that viscous effects in the inner layer depend only on the local turbulence Reynolds number, $\sqrt{\tau/\rho} y/\nu$. There is just one difficulty: the values of A^+ in the pipe and in the duct are different for a given value of $\partial\tau^+/\partial y^+$. It can be seen from figure 4 that the values of A^+ that give the best fit to Patel and Head's duct measurements agree fairly well with the empirical relations suggested by several authors for other plane flows. The pipe measurements stand apart. The only possible conclusion is that transverse curvature affects the viscous sublayer even when the sublayer thickness (to $y^+ = 30$) is less than 10 percent of the radius. The sense of the difference between pipe and duct indicates that concave transverse curvature (as in the streamwise corner between a blade and a hub) tends to suppress the Reynolds stress in the sublayer. One hopes that the effect of transverse curvature is confined to the viscous sublayer. If curvature also affects the inner layer ($30\nu/u_\tau < y < 0.2R$, for instance) then the coincidence of the logarithmic laws in pipe and duct (at high Reynolds numbers) takes a great deal of explaining. However, it is very difficult to see why transverse curvature *should* affect the viscous sublayer and not the fully turbulent flow. The small amount of data on the axial flow over a cylinder (convex curvature) shows that the effect on A^+ is of sign opposite to that in a pipe, as one would expect; the effect may, however, be smaller. Again, the inner layer seems unaffected by curvature as such.

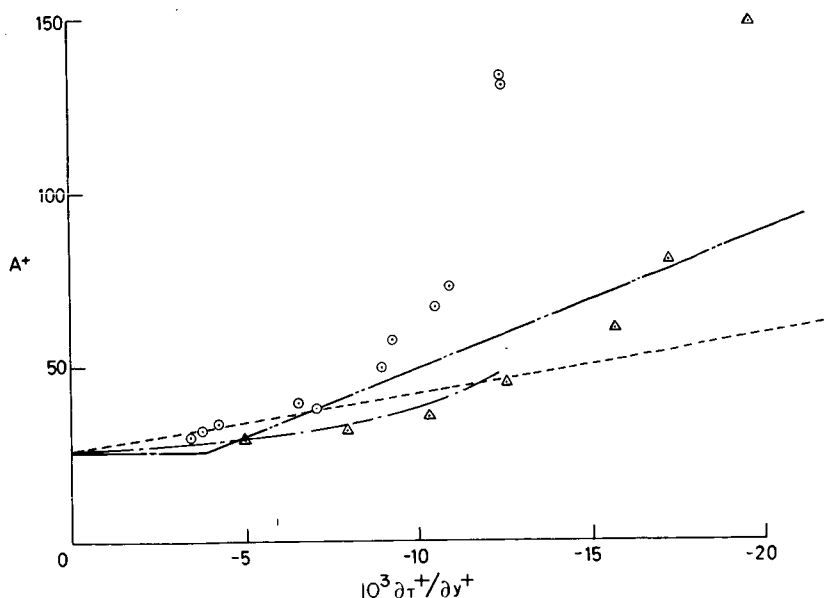


FIGURE 4.—Variation of best-fit value of Van Driest parameter A^+ (eq. (5)) with dimensionless shear-stress gradient. Data of Patel and Head (ref. 10) for pipe (circled points) and duct (triangular points).

FREE-STREAM "TURBULENCE"

The word "turbulence" appears in quotation marks because the unsteady free stream in a turbomachine consists partly of true turbulence and partly of nonrandom fluctuations caused by the relative motion of the blades and their wakes. The boundary layer beneath any unsteady stream with typical wavelengths large compared to the boundary layer thickness can be treated by simple extensions of calculation methods for steady flows. If one ignores streamwise "history" effects on the turbulent shear stress in steady flow one can evidently ignore timewise history effects of similar magnitude in unsteady flow. However, it is more satisfying to analyze the effects of unsteadiness by extending a steady-flow method that takes streamwise history into account. As an example, the calculation method we developed at NPL uses an empirical equation for the rate of change of turbulent shear stress along a streamline, expected to be valid if that rate of change is not too large compared to the rates of production or dissipation of turbulent energy. Exactly the same equation can be used in unsteady flow (ref. 15) simply by noting that the rate of change of shear stress along a streamline now contains $\partial \tau / \partial t$ as well as

spatial derivatives. The equation is again expected to be valid if the total (temporal and spatial) rate of change of shear stress along a streamline is not too large compared to the rate of production or dissipation of turbulent energy. The equations for steady and unsteady flow are identical if we use the notation D/Dt for rate of change along a streamline. Of course, the numerical calculations are more difficult in unsteady flow, but no new physics is required. The same simple extension could be made for any other method using a differential equation for shear stress. Figure 5 shows some calculations by the NPL method for the simple case of an unsteady flow over an infinite plate (independent variables y, t only). This graph is included to show that time-dependent flows can behave rather unexpectedly. Both flows have the same value of $\partial U/\partial t$ as seen by an observer moving with the free stream but the pressure rise to separation is very different. Dr. V. C. Patel of the Lockheed-Georgia Company has programmed our method for two-dimensional unsteady flow (independent variables x, y, t ; see ref. 15a).

If the wavelength of the free-stream fluctuations is of the same order as the wavelength of the boundary-layer turbulence, rates of change of

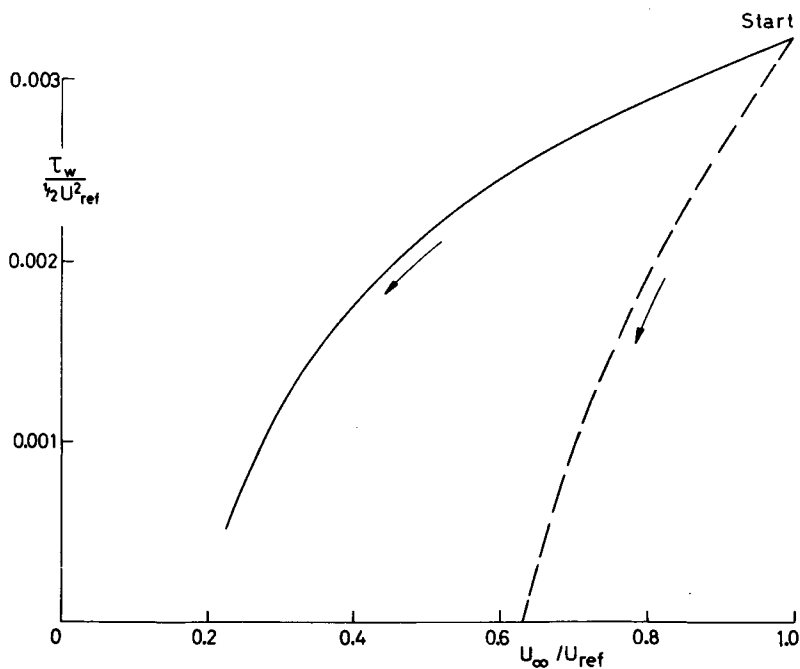


FIGURE 5.—Comparison of calculations for time-dependent and space-dependent boundary layers with the same free stream acceleration. Solid curve: time-dependent, $U = U_{ref} \exp(-0.25 U_{ref} t/l)$. Dashed curve: space-dependent, $U = U_{ref} (1 - 0.25 x/l)$.

shear stress along a streamline may be large; the above-mentioned calculation method then fails, like all others, and we have to consider the effects of the free-stream fluctuations on the turbulence structure. At present, we have no quantitative ideas about what happens. Qualitatively, we can see that free-stream fluctuations will further distort the irregular "superlayer" boundary between the boundary-layer turbulence and the external stream, leading to increased entrainment, so that there may be some similarity between the effects of small-scale free-stream turbulence and viscous effects in the outer layer at low Reynolds numbers. (Coles' paper shows that the two produce similar changes in the mean velocity profile.) We need more data. For recent work see refs. 16a to 16e, D-18 and D-19.

INTERACTING SHEAR LAYERS

A problem which is harder to solve than that of free-stream turbulence, but perhaps easier to study, is that of interaction between two turbulent shear layers. Examples include the flow near the centerline of a wake or jet or in the entrance region of a duct; the effect of a wing or blade wake on a slotted flap or a following blade, respectively; the wall jet; and the boundary layer in a streamwise corner. The last-named is a three-dimensional problem and, on a fundamental level, much harder than the others, although it may be easier to cope with empirically than the blade-wake problem because fewer parameters are involved.

As an example, let us look at the simplest problem, the flow near the centerline of a "two-dimensional" duct. As the growing boundary layers on the two walls approach the centerline, occasional tongues of turbulent fluid from one boundary layer, bearing (say) a positive shear stress, will cross the centerline and enter the negative velocity gradient of the other boundary layer. The turbulent fluctuations in the tongue will be attenuated because the rate of production of turbulent energy (shear stress times velocity gradient) in the tongue will be negative. As the boundary layers continue to grow, the number and intensity of these "frontier violations" will increase, significantly altering the turbulence in the outer part of each boundary layer. Not only will the turbulent intensities and shear stress be changed, but typical eddy length scales and structural coefficients like the energy diffusion coefficient, $\overline{q^2}v/(\overline{q^2})^{3/2}$, may also change. The effects of the interaction on the turbulence seem to penetrate as close to the surface as 0.2 of the half-width of the duct and, of course, the interaction eventually stops streamwise change altogether. In an asymmetrical flow (such as a curved duct or a duct with one rough and one smooth wall (ref. 17)), there is a region in which the net shear stress and velocity gradient have different signs and so the net rate of production

of turbulent energy will be negative. This phenomenon has attracted some attention, but it is merely an overt sign of what goes on in secret in a symmetrical interaction.

The main question to be answered is, "Does the interaction seriously change the turbulence structure?"—meaning the dimensionless properties like the diffusion coefficient mentioned above, rather than dimensional quantities like the intensity. If each of the boundary layers that meet to form a duct flow continued to behave like a boundary layer (with an effective thickness somewhat larger than the half-width of the duct) then we could calculate the flow development by using ordinary boundary-layer methods on each, predicting the two shear stress profiles separately (but combining them in the mean motion equation). We cannot possibly hope that matters are as simple as this, but (refs. 17a and 17b) the effects of the interaction on the turbulence structure are small enough for multiple shear layers to be treated as separate layers, slightly modified by their

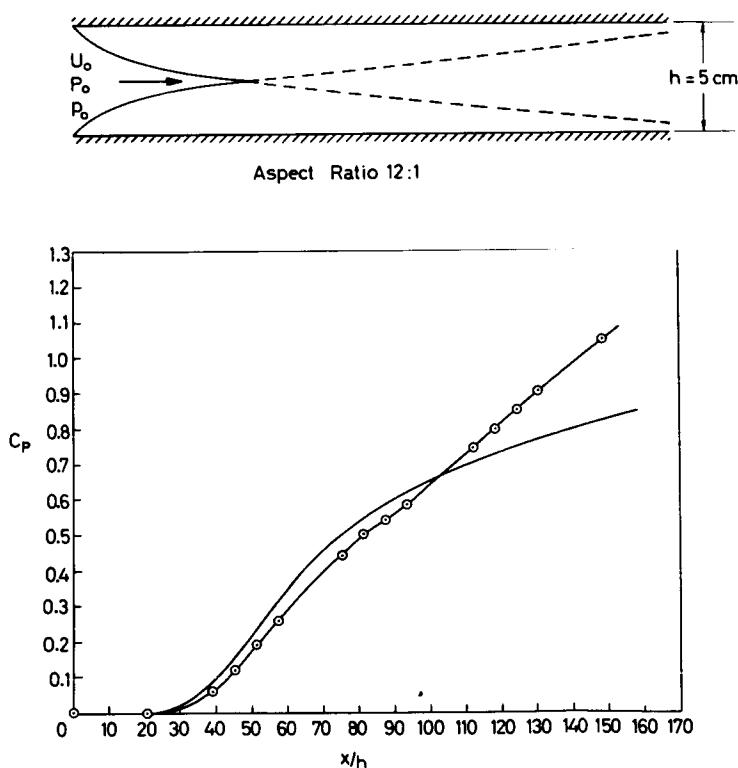


FIGURE 6.—Total-pressure loss on centerline of developing duct flow. C_p is total-pressure loss coefficient, $(P_0 - P)/(P_0 - p_0)$; measurement is denoted by the circled points, and extrapolation of boundary-layer growth, neglecting interaction, is denoted by the solid curve.

neighbors, rather than as one impossibly complicated mess. A simple piece of evidence for this is shown in figure 6. Here we have plotted the total-pressure deficit on the centerline of a duct, compared with *twice* the total-pressure deficit that would occur at the same distance from the surface in a boundary layer growing unimpeded (i.e., we are adding together the total-pressure deficits in the two boundary layers, pretending that they do not interact at all). Of course, the two curves in figure 6 diverge eventually because the boundary layers do *not* continue to grow unimpeded, but at least we can see that nothing very spectacular happens to the shear-stress-producing part of the turbulence when the boundary layers meet. A similar behavior was found by Knystautas (ref. 18) in interfering jets. My doctoral student, Mr. R. B. Dean, whose results are shown in figure 6, is now looking at the details of the turbulence in the interaction region of the duct (ref. 19).

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DISCUSSION

J. H. HORLOCK (Cambridge University): As usual, Mr. Bradshaw has provided a stimulating contribution in his excursion into turbomachinery fluid mechanics. At Cambridge, we are fortunate to have close collaboration with his group at Imperial College, so we have been given early warning on several of the points he raises and have had early opportunity to use his latest ideas.

Mr. Bradshaw's special points about boundary layers in turbomachines are not equally important for the thin boundary layers growing on successive profiles and the thicker annulus wall layers growing continuously through the (axial) machine. It appears, mainly from calculations, that three-dimensional effects on profile boundary layers are small (see ref. D-1), but they are obviously large in annulus wall layers, and this is where the main emphasis of our work at Cambridge lies. Camber effects on turbulence structure are large on profile layers but small on wall layers (δ/R is small). Strong accelerations apply equally to both, but rapid changes of direction are more important near the annulus walls. Low-Reynolds-number effects apply to the profiles but not the walls and high free-stream "turbulence" to both types of layer. I should like to comment on several of these effects in more detail.

(1) I think that the interaction of the "inviscid" secondary flows (which are controlled largely by entry shear and blade geometry) with the viscous regions are probably more important than the "isotropy" of eddy viscosity or mixing length. Perkins (ref. D-2), studying the three-dimensional boundary layer just outside a corner boundary layer, finds the isotropic eddy viscosity concept quite reasonable, but I am bound to say that in our three-dimensional integral methods of calculating the annulus boundary layers where we use the Prandtl-Mager model for the crossflow (thereby avoiding the use of mixing length or eddy viscosity) it is the crossflow that is poorly predicted (Horlock and Hoadley, ref. D-3).

(2) The camber effects are undoubtedly important, but in direct measurements of shear stress in three-dimensional boundary layers developing over cylindrical hubs of diffusers Hughes (ref. D-4) finds no evidence *as yet* that shows β is as large as 7. Hughes has also devised an experiment in which we can compare directly shear stresses with and without Coriolis effects.

(3) Rapid accelerations worry us a great deal in our boundary-layer calculations. Applying the integral method of reference D-3 to the flow through a set of inlet guide vanes of large camber, we find that the stream-wise boundary layer (assumed to be described by a Coles profile) is subject to such an enormous acceleration that the value of π becomes negative. This is not an unknown phenomenon in conventional aerodynamics, but if $\pi < -1$, then the method loses validity. Surprisingly, the negative π profiles compare reasonably well with experiment, with a "hump" in the velocity profile. At present, Marsh and Daneshyar at Cambridge University are planning an experiment in which we can provide rapid acceleration, but not through turning the flow—simply by measuring the wall boundary-layer flow through thick uncambered blades. We are thus attempting to separate the acceleration from the three-dimensional effects due to turning.

(4) The R_θ effects are intriguing and undoubtedly relevant to profile boundary layers. However, I think the main problem relating to these layers is the question of transition, especially in an unsteady environment or one with high free-stream turbulence. At Cambridge, Evans has shown that when the latter is increased, Thwaite's prediction of the laminar separation point becomes pessimistic.

(5) In our group, Daneshyar and Muggleston have also programmed the unsteady Bradshaw method and we look forward to comparing results with Patel. We have also developed an unsteady version of the integral method described in reference D-5 for comparison with the more accurate Bradshaw method. This is a general point of some importance. Our philosophy is that it is unlikely that the Bradshaw type of calculation will be used directly in turbomachine work because it requires even further complications in these real situations (effect of body forces, unsteadiness, three-dimensionality, etc). We feel that simple integral methods may still have uses if they compare reasonably with Bradshaw's method in some trial situations.

(6) The interaction of the shear layers is a new one which I have not thought of before, but obviously of importance in turbomachines, particularly in the effect of a blade wake on a following row. Our only contribution here is the work of Perkins referred to above. Here, the lesson appears to be that if the Reynolds stress distributions can be described with fair accuracy (by correlation of experimental data) the overall parameters such as displacement and momentum "areas" may be predicted and the nature of the secondary flows explained.

I should like to congratulate Mr. Bradshaw on his stimulating paper and look forward to further contributions from him in this area. Perhaps I might emphasize once again that boundary-layer phenomena in turbomachines are also closely related to "inviscid" phenomena such as

secondary and tip clearance flows, but it is very useful to have a new "viscous" recruit to the internal flow area.

H. McDONALD (United Aircraft Research Laboratories): In addition to the seven points of difference between conventional airfoil boundary layers and those encountered on the blades of turbomachinery listed by Mr. Bradshaw, I would like to add two additional points, one on transition and one on separation, and comment upon the low Reynolds number remarks made by Mr. Bradshaw as point (5).

First, experimental evidence obtained by the Pratt & Whitney Division of United Aircraft (ref. D-6) shows clearly that at the low Reynolds numbers typical of turbine blade operation the effect of the high free-stream acceleration on the suction side is to inhibit the transition to fully turbulent flow, in spite of the very high levels of free-stream turbulence induced by the upstream combustion process. In figure D-1, a typical result from reference D-6 is reproduced, illustrating the foregoing remark, and it can be clearly seen that over most of the chord the boundary layer is transitional. It is apparent from figure D-1 that some means of predicting the behavior of transitional boundary layers must be evolved before the heat transfer to the suction side of a turbine can be predicted to an acceptable level of engineering accuracy.

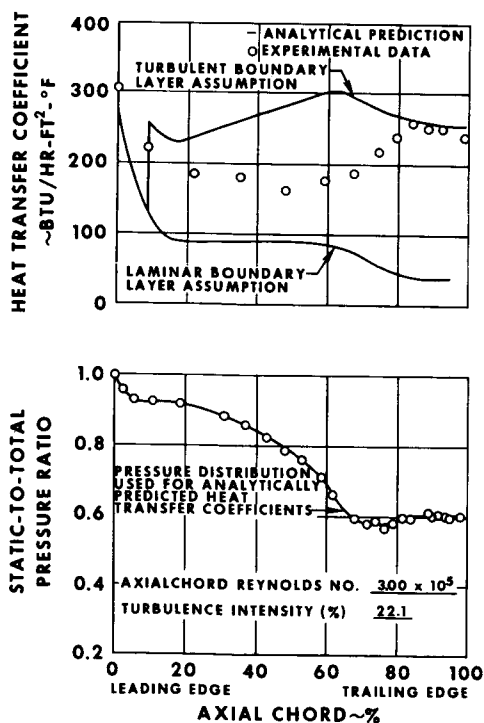


FIGURE D-1.—Experimental heat-transfer coefficient and pressure distribution for a typical turbine vane. (United Aircraft Research Laboratories)

Second, as a result of the low Reynolds numbers commonly encountered in turbomachinery, boundary-layer separation occurs very readily. Conventional separation near the trailing edge usually occurs without reattachment of the boundary to the airfoil surface and often such a separation generates large increases in the loss level. Near the leading edge, separation is usually followed by subsequent reattachment, forming a separation bubble, probably with comparatively little change in the overall loss level. Sometimes, in a poorly designed, highly loaded airfoil, separation can occur near the gauge point. Separation bubbles can, of course, give rise to unexpected Reynolds number effects and, in addition, usually play havoc with boundary-layer prediction methods. Since, with an arbitrary, prescribed pressure distribution, the boundary-layer equations of motion are, in fact, singular at a point of zero wall stress, the more accurately a prediction procedure treats the boundary-layer equations the more likely it is to fail at a separation point. The only rigorous course of action appears to be to use the full Navier-Stokes equations in the region of the bubble. Recently, a study has been completed at United Aircraft Research Laboratories utilizing a finite difference procedure for computing the Navier-Stokes equations in the region of the bubble (ref. D-7). These calculations have shown a considerable upstream influence of the separation process. Future work in this area will involve incorporation of a turbulence model into the procedure.

Concerning the direct effect of Reynolds number, the structure of the low Reynolds turbulent boundary layer has been evaluated recently at United Aircraft Research Laboratories from the equations of mean motion (ref. D-8) using Coles' velocity profile (ref. 11). It was found that the disappearance of the wake component as the Reynolds number is reduced, observed by Coles, is consistent with a large increase in normalized eddy viscosity or mixing length. (At $R_\theta = 500$ the normalized mixing length was double what it was at $R_\theta = 5000$.) Use of a normalized eddy viscosity or mixing length which does not vary with Reynolds number causes the skin friction to be underpredicted by about 10 percent. The variation of eddy viscosity derived from Coles' profile was in fair agreement with the suggestion of Herring and Mellor (ref. 13).

Finally, although the blade profile boundary layers seem amenable to treatment by modest developments of conventional boundary methods, the annulus boundary layer seems particularly troublesome. In addition to being strongly three-dimensional, as Mr. Bradshaw points out, the annulus boundary seems capable of interacting strongly with the "free" stream and distorting the inviscid flow by an appreciable amount. As is well known, this coupling between the annulus boundary layer and the inviscid flow results in both displacement thickness effects and secondary flows, and it would be mandatory to take these effects into account in any stage loss calculation procedure based on boundary-layer theory.

G. L. MELLOR (Princeton University): Professor Bradshaw's paper is an interesting one and I find myself in general agreement with the conclusions expressed there. I will, therefore, attempt to add a corroborative point or two.

(1) Mr. Ronald So of our laboratory has just completed some interesting measurements on wall curvature illustrating the dramatic effect of, for instance, $\delta^*/R \simeq .01$; going from a flat surface to a convex surface, the Reynolds stress in the outer 60 to 70 percent of the layer is virtually "turned off". First indications are that this effect appears to be quantitatively deducible from a Prandtl-Rotta type boundary-layer model.

(2) Mr. Luc Bissonnette of our laboratory has also completed measurements of an axisymmetric boundary layer on a rotating cylinder. We find that the simple eddy viscosities differ by about 30 percent in the axial and circumferential directions.

(3) Characterization of the inner viscous layer is functionally equivalent using either Van Driest's formula (eq. (5)) or the one we used,

$$\frac{\nu_t}{\nu} = \frac{\chi^4}{\chi^3 + (6.9)^3}$$

where $\nu_t = -\overline{u'v'}/(\partial\bar{u}/\partial y)$ and $\chi \equiv u(y/\nu)\sqrt{\tau/\rho}$. When Van Driest's function is mapped onto ours or vice versa, the detailed distributions do differ in what one would think to be an unimportant way. However, in the case of wall suction or blowing it appears that A^* in Van Driest's equation must be adjusted as a function of blowing rate, whereas our formula does not seem to require adjustment. This must currently be considered fortuitous; however, I mention it since the same situation might prevail with regard to the effect of $\partial\tau^+/\partial y^+$.

Speaking in general terms, it is my feeling that the most important turbulent boundary layers in a turbomachine are the annulus wall layers, which probably defy description even in terms of the seven attributes listed in the author's abstract.

J. M. ROBERTSON (University of Illinois): This review, and especially the quantification embodied in table I, indicates serious additional problems imposed on boundary-layer type flow analyses when turbomachinery applications are involved. For several years we have had a number of these under study, as motivated by machinery interests, together with the unlisted one of the turbulent near wake of blades. The present remarks pertain to items (3), (5), and (6) of table I of the authors' listing.

The discussion of streamline curvature effects suggests an increased likelihood of separation on convex surfaces due to a reduction in turbulent shear stress. A study of the boundary layer on the upper surface of a simulated turbomachinery blade by Dr. R. C. Hansen (ref. D-9) indi-

cated turbulent separation much ahead of the locale predicted by analysis. It has been supposed that the breakdown in analysis may be due to a variation in pressure across the layer under appreciable streamline curvature, as such variation is not ordinarily included in boundary-layer analyses. Other studies in our laboratory (refs. D-10 and D-11) of flows approaching turbulent separation have evidenced such a breakdown in the basic boundary-layer premise of constant pressure across the layer. An experiment, in which the turbulent layer developed along a flat plate is sent along the outer surface of a circular cylinder, has been set up to study this occurrence under more controlled conditions. Although only preliminary measurements are available, it is found that already at the point of tangency a pressure change of $0.05\rho U^2/2$ occurs across the layer.

A low-Reynolds-number defect in frictional formulation is suggested in table I. To this writer, this appears to be a matter of what C_f formulation is employed; a local friction factor formulation drawn from the Schoenherr average C_f relation does not seem too appropriate. The 1953 relation of D. Ross (ref. D-12)

$$C_f = (4.4 + 3.8 \log R_\theta)^{-2}$$

where $R_\theta = \theta U/\nu$ has been well verified for flat-plate boundary-layer flows and agrees with Cole's tabulation within a few percent. For adverse-pressure-gradient flows, the Ludwig and Tillman formulation in terms of R_θ and H ($H = \delta^*/\theta$, ratio of displacement to momentum thicknesses) is almost universally accepted; however, several years ago our calculations suggested that this yielded poor values at low Reynolds numbers. The predictions were checked in the flat-plate case where H is well established (refs. D-12 and D-13) as a unique function of R_θ . The Ludwig and Tillman formulation was found to be 19-percent low at $R_\theta = 300$, about the smallest turbulent-layer Reynolds number to be expected. In the spirit of the Ludwig and Tillman formulation, the following expression was developed to circumvent the error.

$$C_f = \frac{\exp [1.8(H_0 - H)]}{(4.4 + 3.8 \log R_\theta)^{-2}}$$

where

$$H_0 = 1.09 + \frac{0.40}{0.72 \log R_\theta - 1.0}$$

is the flat-plate turbulent-layer shape factor (ref. D-14) formulation based on a large number of observations (refs. D-12 and D-13).

For some time, we have been studying the effect of free-stream turbulence on the turbulent boundary layer. This has the effect of increasing the lateral momentum transfer, thus making the velocity profile more uniform over most of the layer; the shape factor H is thus reduced in

magnitude, while G ($G = \delta/\theta$, ratio of layer disturbance thickness (locale of $0.99U$) to momentum thickness) and C_f are increased by the turbulence. In an attempt to quantify these occurrences beyond the rather scattered data available in the literature, we have been studying the turbulent layer on a flat plate downstream of various turbulence-producing grids (up to 10-percent turbulence intensity). Comparison of the changes in the boundary-layer parameters versus those expected in nominally low-level turbulence indicates appreciable changes at intensities up to about 5 percent and then some leveling off. The author's estimate of a 3-percent turbulence level producing a 10-percent increase in C_f is rather well verified; at this level, the shape factor H is reduced by some 4 percent.

A. S. MUJUMDAR (Carrier Corporation): I just wish to point out a few recent papers that would complement the excellent review made by Professor Bradshaw.

Chin, Hulschos, and Hunnicutt (ref. D-15) have reported on their experimental investigation of the effect of lateral curvature on the characteristics of turbulent boundary layers, while Willmarth and Chi (ref. D-16) more recently considered the effect of transverse curvature on wall pressure fluctuations and the turbulence microstructure. From the spectral and cross-correlation measurements of reference D-16, it appears that the turbulence structure of the boundary layer, primarily that of the viscous sublayer, is affected by the transverse curvature. An interesting analytical study of the effect of longitudinal (streamwise) surface curvature on the turbulent boundary layer has been reported recently by Dr. Neal Tetervin (ref. D-17). Although his equations do not give accurate quantitative results, they show that if the concave curvature increases the shear sufficiently, separation is delayed despite the boundary layer thickening. The converse is also true; i.e., convex curvature hastens separation even though the boundary layer is thinned. An important indication from his calculations, which is of special interest to turbomachine designers, is that when the curvature increases in the streamwise direction, separation is hastened on a surface of concave curvature and is delayed on a surface of convex curvature.

Regarding the effect of free-stream turbulence on turbulent boundary layers, there has been some work in this area since the pioneering work of Kline et al. Kestin and his co-workers at Brown University (ref. D-18) and Junkhan and Serovy (ref. D-19) at the Iowa State University have made some valuable contributions in this area.

BRADSHAW (author): I am grateful to the discussors for their comments. In particular, Professor Horlock's second paragraph clarifies the relative importance of the different effects I mentioned, and all of the discussors mention additional effects.

The paragraph numbers below refer to the numbered list in the Abstract.

(1) The effects of anisotropy of eddy viscosity in three-dimensional flow may well be negligible in flows dominated by pressure gradients or viscous inviscid interactions (see the last sentence of the section on three-dimensional boundary layers); however, if the Reynolds stresses outside the local-equilibrium inner layer are important, anisotropy of eddy viscosity is likely to be important also (see figure 2).

(3) Further work on curved flow by my student, Mr. Ian Castro, agrees with So's work mentioned by Professor Mellor. Figure D-2 (also see figure 3) shows the response of \bar{u}^2 in a suddenly deflected shear layer. The maximum δ/R is about 0.025, about the same as Professor Robertson's and roughly one-third of So's. It is notable that the decrease in \bar{u}^2 (taken as the maximum value at a given station and measured in the direction of the local mean velocity) lags behind the increase in curvature. If the subsequent increase of \bar{u}^2 above its initial value is genuine (and it does not seem to be caused by large-scale unsteadiness of the shear layer), one is led to suspect that the recovery of the turbulence from its partly damped state resembles laminar-turbulent transition, in which intense well-organized disturbances appear. I hope the experiments mentioned by the

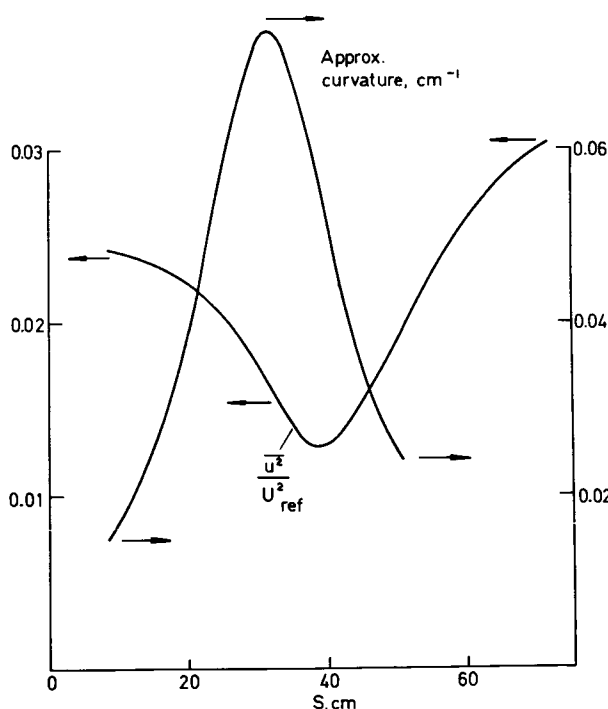


FIGURE D-2.—Deflected shear layer (Castro); see figure 3 of paper.

discussors will help us to represent curvature effects more accurately in shear-layer calculation methods, but I wish to point out that the partly-stabilized state (δ/R of order .01) is more important for blade and airfoil calculation than the "turned off" state, and the latter may not throw much light on the former. Incidentally, I gather that Professor Horlock's student, Mr. Hughes, has now found $\beta \approx 6$ to 7 in a flow with an effective δ/R of about 0.007.

(5) I agree with Professor Robertson that one can find a better C_f law than Schoenherr for low Re ; the point is that Schoenherr is based on the log law and the defect law, and so a failure of Schoenherr implies a failure of the log law or the defect law, which merits investigation in its own right.

As mentioned in the paper, I am fairly confident that, at least in moderate pressure gradients, it is only the outer layer (*defect law*) that alters, owing to viscous effects.

Dr. J. E. Green of R.A.E., Bedford, has also deduced mixing length and eddy viscosity in the outer layer from Coles' low Re profiles; his results agree with Mr. McDonald's in showing large changes.

Figure D-3 shows the final results of Dr. David Huffman's data analysis for strong negative shear stress gradients (or pressure gradients); transverse curvature affects the sublayer behavior for a given $\partial\tau^+/\partial y^+$. This result is qualitatively independent of the sublayer model used, so replacement of Van Driest's formula by Professor Mellor's formula would not collapse the curves, though it would be interesting to see if the duct (flat surface) results were better represented by the latter formula than by Van Driest's formula with constant A^+ .

Of course I agree that, as Professor Horlock and Mr. McDonald say, transition is an important low-Reynolds-number effect; I said nothing about it in the paper because I had nothing to say. The effect of turbulence changes at low Re on $\int C_f dx$, integrated from the leading edge to where $Re_\theta = 5000$, is about the same as a change in transition Re_θ from 400 to 300—not negligible compared to the actual uncertainty of transition position.

(6) Professor Robertson's measurements of the effects of free-stream turbulence are very welcome (not only because they confirm my rough estimate). I hope he or others will look at the effects on the turbulence structure of the shear layer itself. In this connection, the latest measurements in a duct entry region by my student, Mr. Bruce Dean, show that the "non-interaction" (superposition) hypothesis works quite well for u^2 , as well as for total pressure P (figure D-4). The difference between this curve for "non-interaction" C_p and that shown in figure 6 results from a change in our method of extrapolating the boundary-layer growth from data upstream of the interaction region; this is surprisingly critical.

I don't think "non-interaction" can be trusted too far (in its simplest form it would imply no effect of free-stream turbulence) but it is a useful concept when dealing with minor interactions.

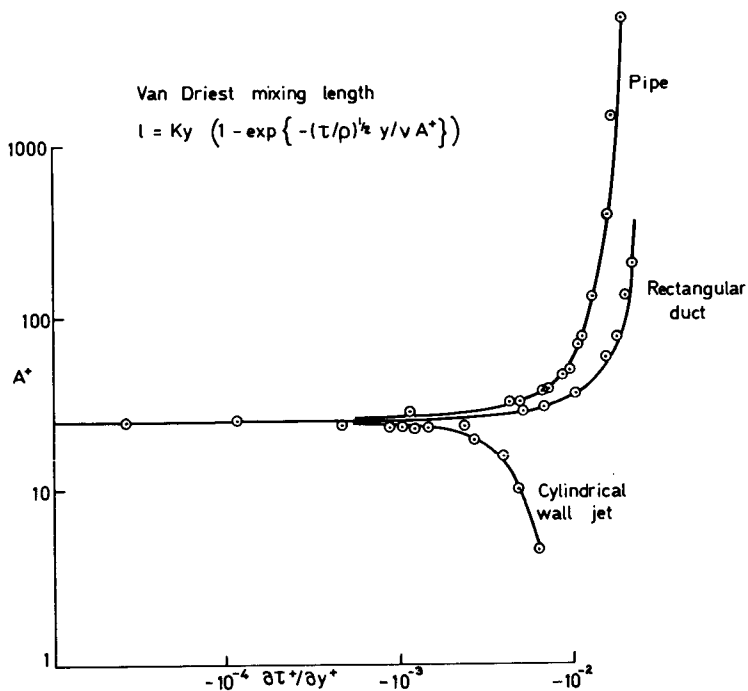


FIGURE D-3.—Sublayer behavior (Huffman); effect of transverse curvature; see figure 4 of paper.

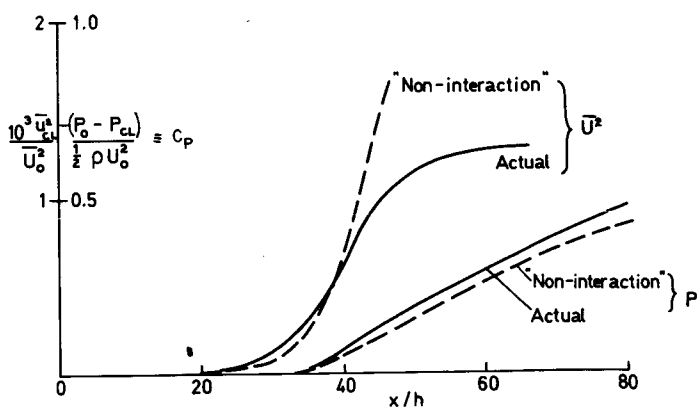


FIGURE D-4.—Duct entry length (Dean); see figure 6 of paper.

The comments about annulus boundary layers made by the discussors emphasize the need to treat viscous/inviscid and viscous/viscous interactions. I hope that people who work on these important problems will not ignore all the knowledge we have gained on thin shear layers; a lot of it will still be usable, provided we have the patience to treat the interaction as an interaction (solution by matching) and do not try to find a complete field solution in one go.

I would change the emphasis of Mr. McDonald's remarks about singularities at separation and say that the less accurately a prediction method treats the physics of the flow the less suitable it will be in calculations of viscous/inviscid interactions. Undoubtedly, one will have to admit elliptic behavior of the pressure and also normal Reynolds stress gradients, but I would not expect any large changes in the turbulence unless the rate of strain changed appreciably from a simple shear.

Finally, I would like to comment on (5) of Professor Horlock's remarks. He rightly points out that real life leads to extra complications in calculation methods and implies that methods like ours, which are already quite complicated, may become intractable, so that integral methods may be preferable. I don't want to argue about the merits of our method in particular, but I think Professor Horlock's "general point" may not be quite as general as he implies.

Let us distinguish between two sorts of turbulence model:

(1) "No history"—local equilibrium between turbulence and mean flow, leading to algebraic equation for shear stress

(2) "History"—differential equation for shear stress and two sorts of solution procedure:

(3) "Integral" method—otherwise, Method of Integral Relations

(4) "Differential" method—finite differences in all independent variables

(3) does not necessarily go with (1) or (4) with (2). At least three people have produced "integral" versions of our calculation method and our "differential" program has built-in options to simulate the input and output of an integral method working on Re_θ , H and C_f . The fastest of the integral versions runs at only five times the speed of the differential program and suffers noticeably from using too crude a velocity profile family. Therefore, I do not think we should reject (4) in favor of (3); computing times for given accuracy will not be an order of magnitude less. Programming time matters only to the originator, not the user. There are some cases where "history" effects are unimportant. In such cases, our method reduces to the mixing-length formula, for which Patankar and Spalding have a program which runs about twice as fast as ours; again, this is not a significant saving unless a vast amount of computing is to be done because the cost of a run is less than a dollar.

Even in three-dimensional (swept wing) flow, the computing time for our method would be only about $2 \times (\text{number of spanwise stations}) \times (\text{two-dimensional computing time})$; roughly the same factor of increase would apply to any other method. The only case which is really daunting is the two-dimensional time-dependent calculation. Unless one uses a method that is implicit in time, solutions for a typical flow whose period of oscillation is many times greater than a typical turbulence time scale are exceedingly lengthy, as Professor Horlock's colleagues have found. With an integral method, one can take larger time steps.

Turning from arithmetical complication to physical complication, the turbulence models used in "history" methods are—or can be—nearer to real life than those used in no-history methods. Therefore, they are *easier* to extend to complicated cases. For instance, extension of our model to time-dependent flow is immediate (see the section of the paper on free-stream "turbulence"); extension to three-dimensional flow requires only a plausible hypothesis based on the observation that turbulence is always three-dimensional; and our present knowledge of the effect of body forces can be included very simply. Therefore, I think that Professor Horlock's view may be slightly colored by his experience with the particularly unpleasant case of unsteady flow, and, while he undoubtedly has a point, it would be a pity if his remarks discouraged turbomachine engineers from trying more complicated turbulence models—they are all much less complicated than turbulence itself.

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